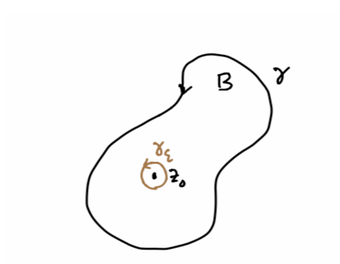
Math 4200-001
Week 7-8 concepts and homework
2.4
Due Friday October 16 at 11:59 p.m.

2.4 2, 3, 5, 7, 8, 12, 16, 17, 18. Hint: In problems 2, 5, 18 identify the contour integrals as expressing a certain function or one of its derivatives, at a point inside γ, via the Cauchy integral formulas for analytic functions and their derivatives.

w7.1 Prove the special case of the Cauchy integral formula that we discuss on Wednesday, in Monday's notes:

If γ is a counter-clockwise simple closed curve bounding a subdomain B in A, with z_0 inside γ, then the important special case of the Cauchy integral formula can be proven with contour replacement and a limiting argument, assuming f is C^1 in addition to being analytic:

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz.$$



w7.2 Prove the positive distance lemma, which we make much use of in proving various theorems: If $K \subseteq \mathbb{C}$ is compact, and if $K \subseteq O$, where O is open, then there exists an $\varepsilon > 0$ such that for each $z \in K$, $D(z; \varepsilon) \subseteq O$. (This is equivalent to Distance Lemma 1.4.21 in the text. See if you can find a proof without looking there first, but in any case write a proof in your own words.)

Math 4200
Monday October 12
2.4 Consequences of Cauchy's integral formula: infinite differentiability of analytic functions; Liouville's Theorem and the fundamental theorem of algebra.

Announcements:

2.4 Recall that before the midterm we proved the Cauchy Integral Formula, which lets us express the values of an analytic function inside closed contours, via a contour integral along these contours:

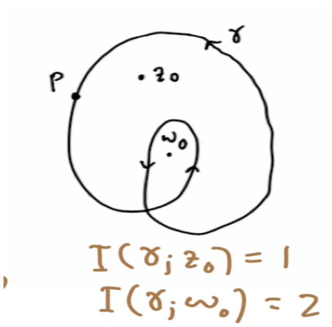
<u>Theorem</u> (Cauchy Integral Formula)

Let $A \subseteq \mathbb{C}$ be open
$f: A \to \mathbb{C}$ analytic
$\gamma: [a, b] \to \mathbb{C}$ a piecewise C^1 closed contour in A that is homotopic (as closed curves in A) to a point. Let $z_0 \notin \gamma([a, b])$.
Then

$$\frac{1}{2 \pi i} \int_{\gamma} \frac{f(z)}{z - z_0}\, dz = f(z_0) I(\gamma ; z_0).$$

The ingredients were:
(1) The fact that index is computed via our "favorite" contour integral integrand:

$$I(\gamma; z_0) = \frac{1}{2 \pi i} \int_{\gamma} \frac{1}{z - z_0}\, dz.$$



(2) The auxillary function

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$$

(3) The deformation theorem for functions closed curves homotopic to points in a domain , applied to the function g.

Example: Let γ be the circle of radius 2 centered at the origin and oriented counterclockwise as usual. Find the value of

$$\int_{\gamma} \frac{\cos(2\,z)}{(z-1)\,\mathrm{e}^{z}}\,dz$$

We stated:
Theorem (Cauchy Integral Formula for Derivatives): Let f be analytic in the open set $A \subseteq \mathbb{C}$, γ a p.w. C^1 contour homotopic to a point in A. Then for z inside γ, every derivative of f exists and may be computed by the contour integral formulas

$$f'(z) I(\gamma; z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta$$

$$f^{(n)}(z) I(\gamma; z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

notice, these are the formulas we get by induction and "differentiating thru the integral sign" :

$$\frac{d}{dz} \frac{f(\zeta)}{\zeta - z} = f(\zeta)(-1)(\zeta - z)^{-2}(-1) = \frac{f(\zeta)}{(\zeta - z)^2}$$

$$\frac{d}{dz} \frac{f(\zeta)}{(\zeta - z)^n} = f(\zeta)(-n)(\zeta - z)^{-n-1}(-1) = n \frac{f(\zeta)}{(\zeta - z)^{n+1}}.$$

So, when can you justify this operation of differentiating thru the integral sign? That's an analysis question!

Analysis answer! General setup: Let γ as usual and

$$G(z) := \int_\gamma g(z, \zeta)\, d\zeta.$$

(For our current needs we will be using the special cases

$$g(z, \zeta) = \frac{f(\zeta)}{(\zeta - z)^n})$$

By linearity of integration,

$$\frac{G(z+h) - G(z)}{h} = \int_\gamma \frac{g(z+h, \zeta) - g(z, \zeta)}{h}\, d\zeta.$$

We wish to know general conditions under which these contour integrals of difference quotients converge to

$$\int_\gamma \frac{\partial}{\partial z} g(z, \zeta)\, d\zeta$$

as $h \to 0$. We certainly need that $g(z, \zeta)$ be complex differentiable in the z variable. Then the following suffices: Suppose the difference quotients converge uniformly (with respect to $\zeta \in \gamma[a, b]$) to $\frac{\partial}{\partial z} g(z, \zeta)$. In other words,

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } \forall\, \zeta \in \gamma[a, b]$$

$$|h| < \delta \Rightarrow \left| \frac{g(z+h, \zeta) - g(z, \zeta)}{h} - \frac{\partial}{\partial z} g(z, \zeta) \right| < \varepsilon.$$

If this uniformity condition holds, then

$$|h| < \delta \Rightarrow \left| \frac{G(z+h) - G(z)}{h} - \int_\gamma \frac{\partial}{\partial z} g(z, \zeta)\, d\zeta \right|$$

$$\leq \int_\gamma \left| \frac{g(z+h, \zeta) - g(z, \zeta)}{h} - \frac{\partial}{\partial z} g(z, \zeta) \right| |d\zeta| < \varepsilon \cdot \text{length}(\gamma),$$

which implies

$$G'(z) = \int_\gamma \frac{\partial}{\partial z} g(z, \zeta)\, d\zeta\ .$$

So, when can we verify the uniformity condition from the previous page?

$$\forall\, \varepsilon > 0\ \exists\ \delta > 0 \text{ such that } \forall\ \zeta \in \gamma[a, b]$$

$$|h| < \delta \Rightarrow \left| \frac{g(z+h, \zeta) - g(z, \zeta)}{h} - \frac{\partial}{\partial z} g(z, \zeta) \right| < \varepsilon.$$

Estimate, assuming $g(z, \zeta)$ is analytic in the $z-$variable and using e.g. line segment contours from z to $z+h$:

$$\frac{g(z+h, \zeta) - g(z, \zeta)}{h} = \frac{1}{h} \int_{z \to z+h} \frac{\partial}{\partial w} g(w, \zeta)\, dw$$

$$= \frac{1}{h} \int_{z \to z+h} \frac{\partial}{\partial z} g(z, \zeta) + \left(\frac{\partial}{\partial w} g(w, \zeta) - \frac{\partial}{\partial z} g(z, \zeta) \right) dw$$

$$= \frac{h}{h} \frac{\partial}{\partial z} g(z, \zeta) + \frac{1}{h} \int_{z \to z+h} \left(\frac{\partial}{\partial w} g(w, \zeta) - \frac{\partial}{\partial z} g(z, \zeta) \right) dw.$$

Regarding the second term as the error term: If for sufficiently small $\rho > 0$, $\frac{\partial}{\partial w} g(w, \zeta)$ is continuous for $(w, \zeta) \in \bar{D}(z; \rho) \times \gamma([a, b])$, then it is uniformly continuous, so

$$\forall\, \varepsilon > 0\ \exists\ 0 < \delta < \rho \text{ such that } \forall\ \zeta \in \gamma[a, b],\ |w - z| < \delta,$$

$$\left| \frac{\partial}{\partial w} g(w, \zeta) - \frac{\partial}{\partial z} g(z, \zeta) \right| < \varepsilon.$$

And in this case, for $|h| < \delta$, the error term is bounded uniformly for $\zeta \in \gamma[a, b]$, by

$$\left| \frac{1}{h} \int_{z \to z+h} \left(\frac{\partial}{\partial w} g(w, \zeta) - \frac{\partial}{\partial z} g(z, \zeta) \right) dw \right| \le \left| \frac{h}{h} \right| \varepsilon = \varepsilon\ .$$

In our applications for the Cauchy integral formulas for derivatives,

$$g(z, \zeta) = \frac{f(\zeta)}{(\zeta - z)^n}$$

$$\frac{\partial}{\partial w} g(w, \zeta) = \frac{n f(\zeta)}{(\zeta - w)^{n+1}}$$

is continuous for $(w, \zeta) \in \bar{D}(z; \rho) \times \gamma([a, b])$ as soon as ρ is small enough so that $\bar{D}(z; \rho) \times \gamma([a, b]) = \emptyset$. (Positive distance lemma).

This finishes the analysis explanation for why the Cauchy integral formulas for derivatives hold.

(From last Wed. notes)

Corollary (*Liouville's Theorem*) Let $f:\mathbb{C}\to\mathbb{C}$ be entire. Suppose f is also *bounded*, i. e. $\exists\ M\in\mathbb{R}$ such that $|f(z)|\le M\ \ \forall\ z\in\mathbb{C}$. Then f is constant.

proof: (It's very very short.)

Let $z\in\mathbb{C}$.

Let γ : circle of rad R ctrd @ z

$$f'(z)=\frac{1}{2\pi i}\oint_{|\zeta-z|=R}\frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta.$$

$$|f'(z)|\le\frac{1}{2\pi}\oint_{|\zeta-z|=R}\frac{M}{R^2}\,|d\zeta|=\frac{1}{2\pi}\frac{M}{R^2}\cdot 2\pi R=\frac{M}{R}.$$

f entire, so R is arb. large.

$\lim_{R\to\infty}$: $\quad |f'(z)|\le 0 \Rightarrow f'\equiv 0 \Rightarrow f$ is const!

Fundamental Theorem of Algebra Let

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

be a polynomial of degree n (scaled so that the coefficient of z^n is 1), with $a_j \in \mathbb{C}$. Then $p(z)$ factors into a product of linear factors,

$$p(z) = (z - z_1)(z - z_2) \ldots\ldots (z - z_n).$$

proof:

- It suffices to prove there exists a single linear factor when $n \geq 1$ since the general case then follows by induction:
  (i) The FTA is true when $n = 1$.
  (ii) If FTA is true for $n - 1$, and if
$$p_n(z) = (z - z_n) p_{n-1}(z)$$
then FTA is true for $p_n(z)$.

- To show that $p_n(z)$ has a linear factor, it suffices to show that $p_n(z)$ has a root, $p_n(z_n) = 0$. This follows from the division algorithm:
$$\frac{p_n(z)}{z - a} = q_{n-1}(z) + \frac{R}{z - a}$$
where R is the remainder. This can be rewritten as
$$p_n(z) = (z - a) q_{n-1}(z) + R.$$
So $p_n(a) = 0$ if and only if $(z - a)$ is a factor of $p_n(z)$.

Then the proof proceeds by contradiction: If $p_n(z)$ has no roots, then $\frac{1}{p_n(z)}$ is entire, and

$$\frac{1}{p_n(z)} = \frac{1}{z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0}$$

$$= \frac{1}{z^n} \frac{1}{\left(1 + \frac{a_{n-1}}{z} + \ldots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)}.$$

Show that $\frac{1}{p_n(z)}$ must be bounded, so by Liouville's Theorem it must be constant. This is a contradiction!